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Exact solution of the deformed biquadratic spin-1 chain

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Abstract. We solve for the spectrum of the deformed biquadratic spin-1 chain by a generalization of the coordinate Bethe ansatz. The model is invariant under the quantum group $U_q sl(2)$ and, depending on the deformation parameter q, it's Hamiltonian may not be Hermitian. We consider several boundary conditions and find in particular that for periodic ones, the complete spectrum of the spin-$\frac{1}{2}$ XXZ model is always contained in the spectrum of the biquadratic model. For free boundary conditions the spectrum is identical to the XXZ model spectrum, only the multiplicities are, of course, different. Bethe states are highest-weight states of the quantum group, except possibly for zero-energy states.

1. Introduction

A model possessing an infinite number of conservation laws is usually referred to as *exactly solvable*, even if there is no way in sight to actually obtain at least a partial solution. Such exactly solvable models can be generated in two dimensions by solving Yang–Baxter, star-triangle or similar cubic equations [1], which guarantee the existence of an infinite number of commuting conserved charges.

One of the favourite methods to apply in such a case is the *Bethe ansatz* (BA). The more sophisticated form is the *algebraic* BA [2]. Here one uses the miraculous fact that the Yang–Baxter equations can be recast in the form of commutation relations for creation and destruction operators with respect to a convenient reference state. It is not known when a particular solution of the Yang–Baxter equations can be used as a starting point for the algebraic BA.

The simplest version is the *coordinate* BA [3] and we will use a generalization of this framework to *solve* the deformed biquadratic spin-1 Hamiltonian subjected to several types of boundary conditions. This Hamiltonian has been obtained as a solution of the Yang–Baxter equations for spin-1 with $U(1)$ invariance [4]. Yet the Yang–Baxter equations do not in this case provide one with commutation relations, so that the algebraic version of the BA is not readily available.

This model has been studied in its undeformed version by Parkinson [5], who obtained the ground state and several low-energy excitations. Barber and Batchelor [6] have shown that the Hamiltonian satisfies the Temperley–Lieb algebra [7]. Klümper has obtained the energy gap using inversion relations [8].

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The deformed model was introduced by Batchelor *et al* [9] who considered spin-1 Hamiltonians invariant under the quantum group $U_q sl(2)$. It is one of the cases which cannot be obtained by a fusion procedure from integrable spin-$\frac{1}{2}$ Hamiltonians [10].

In terms of spin-1 variables $S_k = (S_k^x, S_k^y, S_k^z)$, the deformed biquadratic spin-1 Hamiltonian is given by

$$H(q) = \sum_{k=1}^{N} e_k \tag{1.1}$$

with

$$\begin{aligned} e_k = (S_k \cdot S_{k+1})^2 - 1 - \sinh^2\lambda \left[S_k^z S_{k+1}^z - (S_k^z S_{k+1}^z)^2\right] \\ + \tfrac{1}{2}\sinh\lambda \left[(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y)(S_{k+1}^z - S_k^z) + \mathrm{HC}\right] \\ + 2\sinh^2(\lambda/2) \left[(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y) S_k^z S_{k+1}^z + \mathrm{HC}\right] \\ + \tfrac{1}{2}\sinh(2\lambda) \left[S_k^z S_{k+1}^z (S_{k+1}^z - S_k^z)\right] \end{aligned} \tag{1.2}$$

where $S_k \cdot S_{k+1}$ denotes the standard rotationally invariant scalar product and λ is a coupling constant. In what follows we will also use the deformation parameters q, β and Δ: $\beta = e^{2\lambda}$, $q + q^{-1} = 1 + \beta + \beta^{-1} = -2\Delta$. At $\lambda = 0$, e_k reduces to the isotropic spin-1 biquadratic chain [6, 5].

e_k can be obtained [4] as a solution of the spin-1 representation of the Hecke algebra with only the $U(1)$ symmetry (generated by $S^z = \sum_i S_i^z$). In fact, e_k obeys the more restrictive Temperley–Lieb algebra:

$$\begin{aligned} &e_k^2 = (q + q^{-1}) e_k \qquad e_k e_{k+1} e_k = e_k \\ &[e_k, e_l] = 0 \qquad |k - l| \geqslant 2 \end{aligned} \tag{1.3}$$

and commutes with the quantum group $U_q sl(2)$: the Hamiltonian density is the Casimir operator $(\mathcal{S}_k \cdot \mathcal{S}_{k+1})^2$, where $\mathcal{S}$ is the spin-1 representation of the $U_q sl(2)$ algebra for one site. It actually is the projector on total two-site spin-0.

The paper is organized as follows. In section 2, we develop a Bethe ansatz for periodic boundary conditions—which may include a phase—for the simplest cases. In section 3, we discuss the most interesting sectors with up to two pseudoparticles. The general case, including the ground-state of the anti-ferromagnetic chain, is treated in section 4. Section 5, contains the solution for free boundary conditions, where we verify that the spectrum is identical to one of the XXZ model. We also discuss the status of the Bethe states as quantum-group highest-weight states. Finally the conclusions are reserved for section 6.

2. The coordinate Bethe ansatz

In order to diagonalize H, we proceed as in typical BA applications.

In the basis, where S_k^z is diagonal with eigenvectors $|+,k\rangle, |0,k\rangle, |-,k\rangle$ and eigenvalues $+1, 0$ and -1, e_k is the 9×9 matrix acting in $|a,k\rangle \bigotimes |b,k+1\rangle$, $a, b = +, 0, -$:

$$e_k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

where 0 is the 3×3 zero matrix and U is the 3×3 matrix defined by

$$U = \begin{pmatrix} \beta & \beta^{1/2} & 1 \\ \beta^{1/2} & 1 & \beta^{-1/2} \\ 1 & \beta^{-1/2} & \beta^{-1} \end{pmatrix}. \tag{2.2}$$

Since this Hamiltonian commutes with the total spin operator $S^z = \sum_n S_n^z$, the eigenvalues of the operator $r = N - S^z$ are good quantum numbers. Due to this $U(1)$ invariance, there always exists a reference state $|\Omega\rangle$ satisfying $H(q)\,|\Omega\rangle = E_0\,|\Omega\rangle$, with $E_0 = 0$. We take $|\Omega\rangle$ to be $|\Omega\rangle = \prod_k |+, k\rangle$. This is the only eigenstate in the sector $r = 0$. All other energies will be measured relative to this state. In every sector r there are eigenstates degenerate with $|\Omega\rangle$. They contain r *impurities*. We call impurity a state $|0, k\rangle$ flanked by at least two states $|+, k \pm 1\rangle$. Since $H(q)$ is a sum of projectors on spin zero, these states are annihilated by $H(q)$. In particular, they do not move under the action of $H(q)$, which is the reason for their name.

For example, in the sector $r = 1$, the eigenspace is spanned by the states $|x_{[0]}\rangle = (+ + + \underset{x}{0} + + \ldots)$, $x = 1, \ldots, N$, with a convenient superposition yielding momentum eigenstates for periodic boundary conditions.

2.1. *States containing one pseudoparticle*

Starting with $r = 2$, we encounter states where the $|0, k\rangle$'s occur in neighbouring pairs. They do move under the action of $H(q)$ and mix with states containing one $|-, k\rangle$. Eigenstates are superpositions of $|x_{[-]}\rangle = (+ + + \underset{x}{-} + + \ldots)$ and $|x_{[00]}\rangle = (+ + + \underset{x}{0}0 + + \ldots)$, i.e.

$$|2; \ldots) = \sum_x \{a(x)\,|x_{[-]}\rangle + b(x)\,|x_{[00]}\rangle\} \tag{2.3}$$

where the ellipses stand for parameters on which the eigenvector is to depend.

We will treat periodic boundary conditions maintaining translational invariance in the following sections. They demand $S_{N+l} = S_l$, implying $a(x + N) = a(x)$ and $b(x + N) = b(x)$.

When $H(q)$ now acts on $|2; \ldots)$, it sees the reference configuration, except in the vicinity of x, and we obtain the eigenvalue equations

$$\begin{aligned} \left(E - \beta - \beta^{-1}\right) a(x) &= a(x+1) + a(x-1) + \beta^{1/2} b(x-1) + \beta^{-1/2} b(x) \\ (E-1)\, b(x) &= \beta^{-1/2} a(x) + \beta^{1/2} a(x+1). \end{aligned} \tag{2.4}$$

The translational invariance implies: $a(x) = a\xi^x$ and $b(x) = b\xi^x$. Substituting this into equation (2.4) we get two eigenstates whose energies are

$$\begin{aligned} |2; \theta\rangle_1 &= \sum_x \xi^x \{\Gamma(\xi^{-1})\,|x_{[-]}\rangle + |x_{[00]}\rangle\} \\ E_1 &= 1 + \Gamma(\xi)\Gamma(\xi^{-1}) \end{aligned} \tag{2.5}$$

$$\begin{aligned} |2; \theta\rangle_2 &= \sum_x \xi^x \{|x_{[-]}\rangle - \Gamma(\xi)\,|x_{[00]}\rangle\} \\ E_2 &= 0 \end{aligned} \tag{2.6}$$

where $\Gamma(\xi) = \beta^{-1/2} + \xi\,\beta^{1/2}$. Here $\xi = \mathrm{e}^{\mathrm{i}\theta}$, θ being the momentum determined from the periodic condition selected: $\theta = 2\pi l/N$, with integer l.

We describe this situation by saying that we have two types of pseudoparticles with energies E_1 and E_2. Whereas the pseudoparticle $|2; \theta\rangle_2$ is degenerate with $|\Omega\rangle$, i.e. propagates with energy $E_2 = 0$, the pseudoparticle $|2; \theta\rangle_1$, propagates with energy

$$E_1 = -2\Delta + 2\cos\theta. \tag{2.7}$$

3. The sectors r larger than 2

The situation becomes non-trivial for $r > 2$, where we can have pseudoparticles interacting with isolated impurities and with pseudoparticles. The main result of this paper is to show that $H(q)$ can be diagonalized in a convenient basis and constructed from products of single pseudoparticle eigenfunctions. From this statement, it immediately follows that the eigenvalues of $H(q)$ will be a sum of single pseudoparticle energies,

$$E = \sum_{n=1}^{p} \epsilon_n \{1 + \Gamma(\xi_n)\Gamma(\xi_n^{-1})\} \tag{3.1}$$

where ϵ_n depends on which type of the pseudoparticle we use: $\epsilon_n = 1$ for $E = E_1$ and $\epsilon_n = 0$ for $E = E_2 = 0$. $\xi_n = e^{i\theta_n}$ and θ_n are rapidities still undetermined at this stage.

3.1. One pseudoparticle and impurities

In the sector $r = 3$, besides trivial non-interacting eigenstates (impurities), we have four coupled propagating types of states containing a pseudoparticle and an impurity. We parametrize them as

$$|3; \ldots\rangle = \sum_{x_1<x_2} \{a_1(x_1, x_2)\,|x_1[0], x_2[-]\rangle + b_1(x_1, x_2)\,|x_1[0], x_2[00]\rangle + a_2(x_1, x_2)\,|x_1[-], x_2[0]\rangle + b_2(x_1, x_2)\,|x_1[00], x_2[0]\rangle\}. \tag{3.2}$$

For even N, the parity of the impurity position is conserved, being either even or odd. This is actually a consequence of $[H(q), \sum(-1)^k (S_k^z)^2] = 0$ for even N.

Since the eigenstates in this sector consist of a pseudoparticle interacting with an isolated impurity, the eigenvalues for such states are parametrized as in equation (2.7).

Periodic boundary conditions require that

$$\begin{aligned} a_1(x_1, x_2) &= a_2(x_2, N + x_1) \qquad & b_1(x_1, x_2) &= b_2(x_2, N + x_1) \\ a_2(x_1, x_2) &= a_1(x_2, N + x_1) \qquad & b_2(x_1, x_2) &= b_1(x_2, N + x_1) \end{aligned} \tag{3.3}$$

and translational invariance specifies

$$a_i(x_1, x_2) = \xi^{x_1} a_i(n) \qquad b_i(x_1, x_2) = \xi^{x_1} b_i(n) \qquad i = 1, 2 \tag{3.4}$$

where $n = x_2 - x_1$ and $\xi = \xi_1\xi_2$. Periodic boundary conditions imply $\xi^N = 1$.

Let us take the block $\epsilon_1 = 1$. We try to build these eigenstates out of translationally invariant products of one-pseudoparticle excitation with parameter ξ_1 and one-impurity with parameter ξ_2 and weight functions $D_i(x_1, x_2)$. Using equations (2.5), (2.6), we may write

$$|3; \xi_1, \xi_2\rangle = \sum_{x_1<x_2} \left\{ D_1(x_1, x_2)(\Gamma(\xi_2^{-1})\,|x_1[-]\rangle + |x_1[00]\rangle)\,|x_2[0]\rangle + D_2(x_1, x_2)\,|x_1[0]\rangle\,(\Gamma(\xi_2^{-1})\,|x_2[-]\rangle + |x_2[00]\rangle) \right\}. \tag{3.5}$$

Comparing this with equation (3.2) and using equations (3.4), we have the parametrization

$$\begin{aligned} &a_i(x_1, x_2) = \xi^{x_1} D_i(n)\Gamma(\xi_1^{-1}) \qquad b_i(x_1, x_2) = \xi^{x_1} D_i(n) \qquad i = 1, 2 \\ &D_1(n) = \xi^n D_2(N - n). \end{aligned} \tag{3.6}$$

$H(q)$ may now be applied to $|3; \xi_1, \xi_2\rangle$ to obtain a set of coupled equations for $a_i(x_1, x_2)$ and $b_i(x_1, x_2)$, $i = 1, 2$. Following [5], we split the equations into *far* equations, when excitations do not meet and *near* equations, ontaining terms when they are neighbours.

The *far equations* are

$$(E-\beta-\beta^{-1})a_1(x_1,x_2)=a_1(x_1,x_2-1)+a_1(x_1,x_2+1)$$
$$+\beta^{\frac{1}{2}}b_1(x_1,x_2-1)+\beta^{-\frac{1}{2}}b_1(x_1,x_2)\qquad x_1+2\leqslant x_2\leqslant N+x_1-2 \tag{3.7}$$
$$(E-1)b_1(x_1,x_2)=\beta^{\frac{1}{2}}a_1(x_1,x_2+1)+\beta^{-\frac{1}{2}}a_1(x_1,x_2)\qquad x_1+2\leqslant x_2\leqslant N+x_1-3$$

and a similar set of eigenvalue equations coupling a_2 and b_2.

Eliminating b_1 (b_2) results in the following eigenvalue equations for a_1, provided $E\neq 0, 1$ (and similarly for a_2):

$$(E-1-\beta-\beta^{-1})a_1(n)=a_1(n-1)+a_1(n+1)\qquad 3\leqslant n\leqslant N-3. \tag{3.8}$$

As usual, we now choose to parametrize D_i by plane waves and try as a solution

$$a_1(n)=\Gamma(\xi_2^{-1})\xi_2^n\qquad 3\leqslant n\leqslant N-3. \tag{3.9}$$

Substituting this in equation (3.8), we obtain for the energy

$$E=-2\Delta+2\cos\theta_2 \tag{3.10}$$

where $\xi_2=\mathrm{e}^{\theta_2}$.

To find θ_2 we must consider the *near* equations:

$$\begin{aligned}
&(E-\beta^{-1})a_1(x,x+1)=a_1(x,x+2)+\beta^{-1/2}b_1(x,x+1)\\
&(E-\beta)a_2(x,x+1)=a_2(x-1,x+1)+\beta^{1/2}b_2(x-1,x+1)\\
&(E-2)b_1(x,x+1)=\beta^{1/2}a_1(x,x+2)+\beta^{-1/2}a_1(x,x+1)\\
&\qquad+\beta^{1/2}a_2(x+1,x+2)+\beta^{-1/2}a_2(x,x+2)\\
&(E-2)b_2(x,x+2)=\beta^{-1/2}a_2(x,x+2)+\beta^{1/2}a_2(x+1,x+2)\\
&\qquad+\beta^{-1/2}a_1(x,x+1)+\beta^{1/2}a_1(x,x+2).
\end{aligned} \tag{3.11}$$

It follows that the amplitudes $b_1(x,x+1)$ and $b_2(x,x+2)$ are actually identical. This has to be true, since they multiply the same state.

Using translational invariance and eliminating the amplitudes b_i, the following set of eigenvalue equations is obtained:

$$\begin{aligned}
&(E^2-(2+\beta^{-1})E+\beta^{-1})a_1(1)=(E-1)a_1(2)+\xi^2(a_1(N-1)+\beta^{-1}a_1(N-2))\\
&(E^2-(2+\beta)E+\beta)a_1(N-1)=(E-1)a_1(N-2)+\xi^{-2}(a_1(1)+\beta a_1(2))\\
&(E-1-\beta^{-1})a_1(2)=\beta(E-1)a_1(1)+a_1(3)\\
&(E-1-\beta)a_1(N-2)=\beta^{-1}(E-1)a_1(N-1)+a_1(N-3).
\end{aligned} \tag{3.12}$$

It is easily seen that this system is incompatible with the parametrization (3.9) extended to $1\leqslant n\leqslant N-1$. As we will see, this is a general feature of our model. The usual BA for the wavefunctions cannot be maintained for amplitudes describing neighbouring excitations. We therefore extend equation (3.9) to $2\leqslant n\leqslant N-2$ only, but set

$$\begin{aligned}
a_1(1)&=\Gamma(\xi_2^{-1})\xi_2+\mathcal{F}_{a_1}\\
a_2(1)&=\Gamma(\xi_2^{-1})\xi_1\xi_2^N+\mathcal{F}_{a_2}\\
b_1(1)&=\xi_2+\mathcal{F}_{b_1}\\
b_2(2)&=\xi_1\xi_2^N+\mathcal{F}_{b_2}
\end{aligned} \tag{3.13}$$

with the constants $\mathcal{F}_i$ to be determined. The wavefunctions therefore develop a 'discontinuity'†.

† The quotation marks are to remind us that we are on a discrete lattice.

All our equations are now satisfied, provided the following *consistency condition* holds

$$\xi_2^{N-2}\xi^2 = 1 \tag{3.14}$$

which gives the allowed values of θ_2. The constants $\mathcal{F}_i$ are then given by

$$\mathcal{F}_{a_1} = -\beta^{1/2}\xi_2^N \qquad \mathcal{F}_{b_1} = \xi_2^N \qquad \mathcal{F}_{a_2} = -\beta^{-1/2}\xi_2 \qquad \mathcal{F}_{b_2} = \xi\xi_2. \tag{3.15}$$

These states in the sector $r = 3$ are therefore the eigenstates with eigenvalues given by equation (3.14), where $\theta_2 = (2\pi m \pm 4\pi l/N)/(N-2)$, with l and m integers. Note that if $Nm \pm 2l$ is a multiple of $(N-2)$, we get states degenerate with the state $|2;\theta\rangle$, which lies in the sector $r = 2$. The origin of the factor $(N-2)$ can be understood [5] by saying that a pseudoparticle can propagate past the isolated impurity, but in so doing causes a shift in its position by two lattice sites.

In the sectors $r > 3$ we also find states which consist of one pseudoparticle with parameter ξ_1 interacting with $(r-2)$ impurities with parameters $\xi_i, i = 2, \ldots, \xi_{r-1}$. We parametrize them as

$$\begin{aligned}|r;\xi_1;\xi_2,\ldots\xi_{r-1}\rangle = \sum_{x_1<\ldots<x_{r-1}} \Bigg\{ &\sum_{i=1}^{r-1} a_i(x_1,\ldots x_{r-1}) \\ &\times |x_1[0],\ldots,x_{i-1}[0],x_i[-],x_{i+1}[0],\ldots,x_{r-1}[0]\rangle \\ &+\sum_{i=1}^{r-1} b_i(x_1,\ldots,x_{r-1})\,|x_1[0],\ldots,x_{i-1}[0],x_i[00],x_{i+1}[0],\ldots,x_{r-1}[0]\rangle\Bigg\}. \end{aligned} \tag{3.16}$$

The energy of these states is parametrized as in equation (3.10) and $\xi_1 = e^{i\theta_1}$ satisfies the phase-shift condition (3.14) with $\xi = \xi_1\xi_2\ldots\xi_{r-1}$. It involves only ξ_1 and $\xi_{\text{imp}} = \xi_2\xi_3\ldots\xi_{r-1}$, being therefore highly degenerate. This is to be expected due to the irrelevance of the relative impurity distances, up to jumps of two positions via exchange with a pseudoparticle.

3.2. Sector $r = 4$

This sector contains, in addition to $|4;\xi_1;\xi_2,\xi_3\rangle$ discussed above, states which consist of two interacting pseudoparticles. We seek these eigenstates in the form†

$$\begin{aligned}|4;\ldots\rangle_{\epsilon_1\epsilon_2} = \sum_{x_1<x_2} \{&a(x_1,x_2)\,|x_1[-],x_2[-]\rangle + b_1(x_1,x_2)\,|x_1[-],x_2[00]\rangle \\ &+b_2(x_1,x_2)\,|x_1[00],x_2[-]\rangle + c(x_1,x_2)\,|x_1[00],x_2[00]\rangle\}. \end{aligned} \tag{3.17}$$

Translational invariance now specifies

$$\begin{aligned} a(x_1,x_2) &= \xi^n a(n) & b_1(x_1,x_2) &= \xi^n b_1(n) \\ c(x_1,x_2) &= \xi^n c(n) & b_2(x_1,x_2) &= \xi^n b_2(n) \end{aligned} \tag{3.18}$$

and the periodic boundary conditions require that

$$\begin{aligned} a(n) &= \xi^n a(N-n) & c(n) &= \xi^n c(N-n) \\ b_1(n) &= \xi^n b_2(N-n) & b_2(n) &= \xi^n b_1(N-n) \end{aligned} \tag{3.19}$$

where $n = x_2 - x_1$, $\xi = \xi_1\xi_2$ ($\xi_i = e^{i\theta_i}$, $i = 1, 2$) and the total momentum is $\theta_1 + \theta_2 = 2\pi l/N$, with integer l.

† The functions $b_i(x_1, x_2)$ are not the same as their homonyms used for $r = 3$.

According to equation (3.17), we will parametrize energy as

$$E = \sum_{n=1}^{2} \epsilon_n \left[1 + \Gamma(\xi_n)\Gamma(\xi_n^{-1})\right]. \tag{3.20}$$

Let us take the block $\epsilon_1 = \epsilon_2 = 1$ first. Again, we try to build 2-pseudoparticle eigenstates out of translationally invariant products of one-pseudoparticle excitations at x_1 and x_2 with weight functions $D_i(x_1, x_2)$, $i = 1, 2$:

$$\begin{aligned}|4; \theta_1, \theta_2\rangle_{11} = \sum_{x_1<x_2} \{ & D_1(x_1, x_2)\left(\Gamma(\xi_1^{-1})|x_1[-1]\rangle + |x_1[00]\rangle\right)\left(\Gamma(\xi_2^{-1})|x_2[-1]\rangle + |x_2[00]\rangle\right) \\ & + D_2(x_1, x_2)\left(\Gamma(\xi_2^{-1})|x_1[-1]\rangle + |x_1[00]\rangle\right)\left(\Gamma(\xi_1^{-1})|x_2[-1]\rangle + |x_2[00]\rangle\right)\}. \end{aligned} \tag{3.21}$$

Comparing this with equation (3.17) and using translational invariance equations (3.18), we get

$$\begin{aligned} a(n) &= \Gamma(\xi_1^{-1})\Gamma(\xi_2^{-1})D(n) \\ b_1(n) &= \Gamma(\xi_1^{-1})D_1(n) + \Gamma(\xi_2^{-1})D_2(n) \\ b_2(n) &= \Gamma(\xi_2^{-1})D_1(n) + \Gamma(\xi_1^{-1})D_2(n) \\ c(n) &= D(n) \end{aligned} \tag{3.22}$$

where $D(n) = D_1(n) + D_2(n)$ and the periodic boundary conditions imply $D_2(n) = \xi^n D_1(N - n)$.

Applying $H(q)$ to the state of (3.17), we obtain a set of coupled equations for $a(n)$, $b_i(n)$ and $c(n)$. When the two pseudoparticles are separated, we have the following *far* equations:

$$\begin{aligned} (E - 2\beta - 2\beta^{-1})a(n) &= (1 + \xi^{-1})a(n+1) + (1 + \xi)a(n-1) \\ &\quad + \beta^{-1/2}[b_1(n) + b_2(n)] + \beta^{1/2}[b_1(n-1) + \xi^{-1}b_2(n+1)] \\ &\quad (2 \leqslant n \leqslant N-2) \\ (E - 1 - \beta - \beta^{-1})b_1(n) &= \xi^{-1}b_1(n+1) + \xi b_1(n-1) \\ &\quad + \beta^{1/2}[a(n+1) + \xi^{-1}c(n+1)] + \beta^{-1/2}[a(n) + c(n)] \\ &\quad (2 \leqslant n \leqslant N-2) \\ (E - 1 - \beta - \beta^{-1})b_2(n) &= b_2(n-1) + b_2(n+1) + \beta^{1/2}[c(n-1) + \xi a(n-1)] \\ &\quad + \beta^{-1/2}[c(n) + a(n)] \qquad (3 \leqslant n \leqslant N-3) \\ (E - 2)c(n) &= \beta^{1/2}[\xi b_1(n-1) + b_2(n+1)] + \beta^{-1/2}[b_1(n) + b_2(n)] \\ &\quad (3 \leqslant n \leqslant N-3). \end{aligned} \tag{3.23}$$

We already know them to be satisfied if we parametrize $D_1(n)$ and $D_2(n)$ by plane waves:

$$D_1(n) = \xi_2^n \qquad D_2(n) = \xi_2^N \xi_1^n. \tag{3.24}$$

The real problem arises, of course, when pseudoparticles are neighbours, so that they interact and we have no guarantee that the total energy is a sum of single pseudoparticle energies.

Action of $H(q)$ on the state $|4;\theta_1,\theta_2\rangle_{11}$ gives the following set of *near* equations:

$$\begin{aligned}
(E-\beta-\beta^{-1})a(x,x+1) &= a(x-1,x+1)+a(x,x+2)\\
&\quad +\beta^{-1/2}b_1(x,x+1)+\beta^{1/2}b_2(x-1,x+1)\\
(E-1-\beta)b_1(x,x+1) &= b_1(x-1,x+1)+\beta^{-1/2}a(x,x+1)\\
&\quad +\beta^{1/2}[\,a(x,x+2)+c(x-1,x+1)\,]\\
(E-1-\beta^{-1})b_2(x-2,x) &= b_2(x-2,x+1)+\beta^{1/2}a(x-1,x)\\
&\quad +\beta^{-1/2}[\,a(x-2,x)+c(x-2,x)\,]\\
(E-3)c(x,x+2) &= \beta^{1/2}[\,b_1(x+1,x+2)+b_2(x,x+3)\,]\\
&\quad +\beta^{-1/2}[\,b_1(x,x+2)+b_2(x,x+2)\,]+\beta^{1/2}p(x)+\beta^{-1/2}q(x)\\
(E-\beta)p(x) &= q(x)+\beta^{1/2}c(x,x+2)\\
(E-\beta^{-1})q(x) &= p(x)+\beta^{-1/2}c(x,x+2).
\end{aligned} \tag{3.25}$$

In deriving these equations, two new states made their debut. They arise through the actions of the Hamiltonian on $(+++\underset{x}{0}000+\ldots)$ with coefficient $c(x,x+2)$. We get the new states $(+++\underset{x}{0}+-0+\ldots)$ and $(+++\underset{x}{0}-+0+\ldots)$. We incorporate them with coefficients $p(x)$ and $q(x)$, respectively, in our ansatz equations (3.17). They are eliminated, applying $H(q)$ to them and substituting the result in the equation for $c(x,x+2)$. Using translational invariance we get

$$\begin{aligned}
(E-\beta-\beta^{-1})a(1) &= (1+\xi^{-1})a(2)+\beta^{-1/2}b_1(1)+\beta^{1/2}\xi^{-1}b_2(2)\\
(E-1-\beta)b_1(1) &= \xi^{-1}b_1(2)+\beta^{1/2}[a(2)+\xi^{-1}c(2)]+\beta^{-1/2}a(1)\\
(E-1-\beta^{-1})b_2(2) &= b_2(3)+\beta^{1/2}\xi a(1)+\beta^{-1/2}[a(2)+c(2)]\\
\left(E-3-\frac{\beta+\beta^{-1}}{E-\beta-\beta^{-1}}\right)c(2) &= \beta^{1/2}[\xi b_1(1)+b_2(3)]+\beta^{-1/2}[b_1(2)+b_2(2)].
\end{aligned} \tag{3.26}$$

Here we notice two features implied by equations (3.23), (3.26). First, the wavefunctions again develop a 'discontinuity'. To see this, take the far equation for $c(n=2)$ and subtract from it the near equation for $c(2)$, to get

$$c(2)\,E/(E-\beta-\beta^{-1})=0. \tag{3.27}$$

This requires either $c(2)=0$ or $E=0$, both of them unacceptable in the block $\epsilon_1=\epsilon_2=1$. Thus $c(2)$ in equations (3.26) is not the continuation of $c(n)$ for $n>2$. So, secondly, the choice of equation (3.24), which would be the usual Bethe ansatz, does *not* in fact solve our near equations.

Therefore, we proceed as follows [5]. Let us make a minimal modification, maintaining $a(n)$, $b_1(n)$ and $b_2(n)$, $c(n)$ for $2\leqslant n\leqslant N-2$ and $3\leqslant n\leqslant N-3$, respectively, and leaving $a(1)$, $b_1(1)$, $b_2(2)$ and $c(2)$ as arbitrary constants. This means that we modify the ansatz equations (3.21) only when pseudoparticles do interact. It is convenient to set

$$\begin{aligned}
a(1) &= \Gamma(\xi_1^{-1})\Gamma(\xi_2^{-1})D(1)+\mathcal{F}_a\\
b_1(1) &= \Gamma(\xi_1^{-1})D_1(1)+\Gamma(\xi_2^{-1})D_2(1)+\mathcal{F}_{b_1} \qquad i=1,2\\
b_2(2) &= \Gamma(\xi_2^{-1})D_1(2)+\Gamma(\xi_1^{-1})D_2(2)+\mathcal{F}_{b_2}\\
c(2) &= D(2)+\mathcal{F}_c.
\end{aligned} \tag{3.28}$$

All equations can now be satisfied provided the following consistency condition holds,

$$\xi_2^N=-\frac{\xi_2[(1+\xi^{-1})\xi_2-2\Delta-E]}{\xi_1[(1+\xi^{-1})\xi_1-2\Delta-E]} \tag{3.29}$$

and provided the constants are given by

$$\mathcal{F}_a = \xi^{-1}\mathcal{F}_c \qquad \mathcal{F}_{b_1} = -\beta^{-1/2}\xi^{-1}\mathcal{F}_c \qquad \mathcal{F}_{b_2} = -\beta^{1/2}\mathcal{F}_c \qquad \text{and} \quad \mathcal{F}_c = \frac{D(2)}{\epsilon + 2\Delta}. \tag{3.30}$$

In the block $\epsilon_1 = \epsilon_2 = 1$, this can be rewritten as

$$\xi_2^N = -\frac{1 + \xi - 2\Delta\xi_2}{1 + \xi - 2\Delta\xi_1}. \tag{3.31}$$

This is the same consistency equation one finds for the XXZ model and it reduces to that of [5] in the isotropic limit $\beta \to 1$.

It is easy to verify that this ansatz also holds for the other blocks, when not all $\epsilon_i = 1$. For $E \neq 0$, the consistency condition equation (3.29) remains identical, provided we use the correct energy, $E = E(\epsilon_1, \epsilon_2)$. Therefore, we get different Bethe ansatz equations for every block (ϵ_1, ϵ_2).

For instance, in the block $\epsilon_1 = 1$, $\epsilon_2 = 0$, we have

$$\begin{aligned}|4; \theta_1, \theta_2\rangle_{10} = \sum_{x_1 < x_2} \big\{ & D_1(x_1, x_2)(\Gamma(\xi_1^{-1})|x_1[-1]\rangle + |x_1[00]\rangle)(|x_2[-1]\rangle - \Gamma(\xi_2)\,|x_2[00]\rangle) \\ & + D_2(x_1, x_2)(|x_1[-1]\rangle - \Gamma(\xi_2)|x_1[00]\rangle)(\Gamma(\xi_1^{-1})\,|x_2[-1]\rangle + |x_2[00]\rangle)\big\}. \end{aligned} \tag{3.32}$$

Using translational invariance and equation (3.17), we now get

$$\begin{aligned} a(n) &= \Gamma(\xi_1^{-1})(D_1(n) + D_2(n)) \\ b_1(n) &= -\Gamma(\xi_1^{-1})\Gamma(\xi_2)D_1(n) + D_2(n) \\ b_2(n) &= -\Gamma(\xi_2)\Gamma(\xi_1^{-1})D_2(n) + D_1(n) \\ c(n) &= -\Gamma(\xi_2)(D_1(n) + D_2(n)) \end{aligned} \tag{3.33}$$

unless the pseudoparticles become neighbours. In this case there is a modification analogous to equation (3.28). The amplitudes are given by equation (3.33) with the same constants equation (3.30) added. The consistency condition equation (3.29) becomes $\xi_2^{N-2} = 1$. Or, using periodic boundary conditions, we obtain

$$\xi_1^N \xi_2^2 = 1 \qquad \xi_2^{N-2} = 1 \qquad E = 1 + \Gamma(\xi_1)\Gamma(\xi_1^{-1}). \tag{3.34}$$

Results for the block $\epsilon_1 = 0$, $\epsilon_2 = 1$ are obtained from the block $\epsilon_1 = 1$, $\epsilon_2 = 0$ by the interchange $\xi_1 \leftrightarrow \xi_2$.

Only the case $E = 0$ falls out of line, as we may expect from (3.27). The parametrization in this case is

$$\begin{aligned} a(n) &= D_1(n) + D_2(n) \\ b_1(n) &= -\Gamma(\xi_2)D_1(n) - \Gamma(\xi_1)D_2(n) \\ b_2(n) &= -\Gamma(\xi_1)D_1(n) - \Gamma(\xi_2)D_2(n) \\ c(n) &= \Gamma(\xi_1)\Gamma(\xi_2)(D_1(n) + D_2(n)). \end{aligned} \tag{3.35}$$

The only condition is $\xi^N = 1$. Since either ξ_1 or ξ_2 is a free parameter, this state is degenerate.

4. General r

The generalization to any r is in principle straightforward. Since the Yang–Baxter equations are satisfied, there is only two-pseudoparticle scattering (if we use the S-matrix language). Therefore near equations, where more then two pseudoparticles become neighbours, are not expected to give any new restrictions. The n-pseudoparticle phase shift will be a sum of two-pseudoparticle phase shifts $\Theta(\theta_1, \theta_2)$:

$$\Theta(\xi_1, \xi_2) = -\mathrm{i}\log\left(-\frac{\xi_2}{\xi_1}\frac{(1+\xi^{-1})\xi_2 + 2\Delta - E}{(1+\xi^{-1})\xi_1 + 2\Delta - E}\right) \qquad E \neq 0 \tag{4.1}$$

with $\xi = \xi_1\xi_2$.

In order to convince the sceptical reader, we outline the situation for this case. For example, for three pseudoparticles we get the following eventually problematic situations in which pseudoparticles meet:

$$(\ldots + - - - + \ldots), (\ldots + -00 + \ldots), \ldots, (\ldots 00000 \ldots). \tag{4.2}$$

It could be expected that the meeting of three pseudoparticles may require new modifications of the amplitudes $a(x_1, x_2, x_3), \ldots$ which could eventually modify the consistency conditions. The first collision can be considered as arising either from $(\ldots + - \to - - + \ldots)$ or $(\ldots + - - \to - + \ldots)$. The equations then show that no new constants of the type of equation (3.30) appear in the parametrization of wavefunctions with more than two pseudoparticles. As a matter of fact, the *near* equations, with more then two pseudoparticles meeting, can always be rewritten so that one pseudoparticle is a spectator, leaving only two interacting pseudoparticles. The equations can then be shown to be satisfied, provided equations (3.30) are true.

The BA equations for $E \neq 0$ and no impurities are then given by

$$\xi_a^N = \prod_{b=1, b\neq a}^{r} -\frac{\xi_a}{\xi_b}\frac{(1+(\xi_a\xi_b)^{-1})\xi_a - \epsilon}{(1+(\xi_a\xi_b)^{-1})\xi_b - \epsilon} \qquad a = 1, \ldots, r \tag{4.3}$$

where $\epsilon = E + 2\Delta$. Since $\xi_i^N = \mathrm{e}^{N\theta_i} = 1$, the total momentum is $P = \sum_i^{r/2}\theta_i = 2\pi l/N$, with integer l.

We will now give the results for sector r with p pseudoparticles and $r - 2p$ impurities and $E \neq 0$.

Energy is given by

$$E = \sum_{i=1}^{p} \epsilon_i(1 + \Gamma(\xi_i)\Gamma(\xi_i^{-1}) \tag{4.4}$$

and periodic boundary conditions require $\xi^N = 1$, where $\xi = \xi_1\xi_2\ldots\xi_p\xi_{\mathrm{imp}}$, $\xi_{\mathrm{imp}} = \xi_{p+1}\xi_{p+2}\ldots\xi_{r-p}$.

The parametrization of wavefunctions in each sector is again modified when excitations are neighbours, by adding constants $\mathcal{F}_i$. All of them can be computed in terms of the constants equation (3.15) for impurity–pseudoparticle meeting and equation (3.30) for pseudoparticle–pseudoparticle meeting, using the far equations containing near terms in their right-hand side. The resulting BA-equations are then

$$\xi_a^N\xi_{\mathrm{imp}}^2 = \prod_{b=1, b\neq a}^{r} -\frac{\xi_a}{\xi_b}\frac{(1+(\xi_a\xi_b)^{-1})\xi_a - \epsilon}{(1+(\xi_a\xi_b)^{-1})\xi_b - \epsilon} \qquad a = 1, \ldots, p. \tag{4.5}$$

As for state equation (3.16), all the impurity dependence is lumped into ξ_{imp} implying a large degeneracy.

Again, $E = 0$ is different. We get a hugely degenerate state with the only condition $\xi^N = 1$.

The above results straightforwardly generalize for twisted boundary conditions:

$$S_N^z = S_1^z \qquad S_N^{\pm} = e^{\pm i\Phi} S_1^{\pm}. \tag{4.6}$$

The periodic boundary condition $\xi^N = 1$ is replaced by $\xi^N = e^{4i\Phi}$. In particular, in the block $\epsilon_i = 1$, the BA equations differ from the XXZ BA equations only by the replacement $\Phi \to \Phi/2$.

All this is true for N = odd. For N = even a new feature arises when $r = N$. This is exactly the sector containing the ground state of the antiferromagnetic chain with $H_{\text{anti}}(q) = -H(q)$.

In this case, all equations for the amplitudes $a, b_1, \ldots$ continue to hold, except for the equation for the amplitude of the state $000\ldots000$ (N), like the one for $c(x, x+2)$, $N = 4$ in equations (3.26), which gets modified. There is now an extra pair of zeroes where the Hamiltonian is to be applied—the ones at $x = 1$ and $x = N$—and no states of the type p and q of equations (3.25) are created. The equation for this amplitude, call it c, now becomes

$$\begin{aligned}(E - N)c = {} & (e^{-i\Phi} + \xi + \xi^2 + \ldots + \xi^{N-1})\beta^{1/2} b_1 \\ & + (e^{i\Phi} + \xi + \xi^2 + \ldots + \xi^{N-1})\beta^{-1/2}\xi\, b_2\end{aligned} \tag{4.7}$$

where b_1 (respectively b_2) stands for amplitudes arising from c by replacing a pair of zeroes by $+-$ and $-+$, respectively, and we use twisted boundary conditions $\xi^N = e^{4i\Phi}$. This will affect only the Φ-dependent eigenvalues. The final result is that in this particular sector these modifications can be incorporated into a change of boundary conditions. The previous results (equations (4.5)) are still valid, subject only to the replacement of the boundary angle Φ' with

$$\cos\Phi' = \cos\Phi + \tfrac{1}{2}. \tag{4.8}$$

This kind of equation has already been proposed in [13], where the authors show that the ground-state energy of the biquadratic model with angle Φ equals the ground-state energy of the XXZ model with boundary angle Φ_{xxz}, provided

$$\cos(\Phi_{xxz}/2) = \cos\Phi + \tfrac{1}{2}. \tag{4.9}$$

5. Free boundary conditions

Since the Hamiltonian with free boundary conditions $H_f(q) = \sum_{k=1}^{N-1} e_k$ commutes with the quantum group $U_q sl(2)$, we expect the Bethe states to be highest-weight states of $U_q sl(2)$. This is indeed true for all states, except for some with $E = 0$. As the reasoning follows the same line of thought as for the periodic case, we will only discuss the cases $r = 2$, 3 and $r = 4$, which are the more interesting ones. Exactly as for the periodic case, no new features appear in the higher sectors and so we will not discuss them.

5.1. The sector $r = 2$

For $r = 2$, the eigenstates are of the form

$$|2;\theta\rangle = \sum_{x=1}^{N} a(x)|x[-]\rangle + \sum_{x=1}^{N-1} b(x)|x[00]\rangle. \tag{5.1}$$

When $H_f(q)$ now acts on $|\theta\rangle$, we obtain the eigenvalue equations

$$
\begin{aligned}
&(E-\beta^{-1})a(1) = a(2) + \beta^{-1/2}\,b(1),\\
&(E-\beta-\beta^{-1})a(x) = a(x+1) + a(x-1) + \beta^{1/2}\,b(x-1) + \beta^{-1/2}b(x)\\
&\qquad\qquad (2 \leqslant x \leqslant N-1) \qquad (5.2)\\
&(E-\beta)\,a(N) = a(N-1) + \beta^{1/2}\,b(N-1)\\
&(E-1)\,b(x) = \beta^{-1/2}\,a(x) + \beta^{1/2}\,a(x+1) \qquad (1 \leqslant x \leqslant N-1).
\end{aligned}
$$

We write the solution as:

$$
\begin{pmatrix} a(x) \\ b(x) \end{pmatrix} = \begin{pmatrix} 1 \\ b_0 \end{pmatrix} \xi^x + \begin{pmatrix} a_0' \\ b_0' \end{pmatrix} \xi^{-x} \tag{5.3}
$$

where $\xi = \mathrm{e}^{\mathrm{i}\theta}$ and find from the eigenvalue equations, for the case of $E \neq 0$:

$$
\begin{aligned}
a(1) &= \frac{\xi - 1/\xi}{1+\beta/\xi} \qquad & a(N) &= -\beta\xi^N a(1)\\
a_0' &= -\frac{1+\beta\xi}{1+\beta/\xi} \qquad & b_0 &= \beta^{1/2}/(1+\beta/\xi) = b_0'
\end{aligned} \tag{5.4}
$$

where $\xi^{2N} = 1$, implying $\theta = \pi l/N$ and integer l.

For the case $E = 0$, all equations reduce to

$$
b(x) = a(x)\beta^{-1/2} - a(x+1)\beta^{1/2} \qquad 1 \leqslant x \leqslant N-1. \tag{5.5}
$$

determining $b_0 = (1+\beta\xi)\beta^{-1/2}$, $b_0' = -a_0'$, b_0', leaving a_0' and θ undetermined, revealing the degeneracy of this state.

We note that the eigenvalues given by the above equations are the same as those of the XXZ for the same value of the deformation parameter q [11].

We can easily check, whether these are quantum-group highest-weight states or not. The action of the quantum group raising operator on the state $|x>$ is

$$
\mathcal{S}_+|x\rangle = \hat{q}^{S_z}\ldots\hat{q}^{S_z}S_+(x)\hat{q}^{-S_z}\ldots\hat{q}^{-S_z}|x\rangle. \tag{5.6}
$$

Imposing $\mathcal{S}_+\,|2;\theta\rangle = 0$ yields the constraints

$$
\begin{aligned}
&a(1) + \hat{q}b(1) = 0\\
&a(x) + \hat{q}b(x) + \hat{q}^{-1}b(x-1) = 0 \qquad (2 \leqslant x \leqslant N-1) \qquad (5.7)\\
&a(N) + \hat{q}^{-1}b(N-1) = 0.
\end{aligned}
$$

For $E \neq 0$ these are satisfied, provided $\hat{q} = -\beta^{-1/2}$. For $E = 0$ they are only true for states, whose momentum $\theta = \pi l/N$ satisfies $\cos(l\pi/N) = \Delta$ or

$$
\mathrm{e}^{\mathrm{i}l\pi/N} = q \tag{5.8}
$$

i.e. when q is a root of unity.

5.2. *The sector* $r = 3$

In this sector we have one pseudoparticle at, say, x_2 and one impurity at x_1. In contrast to equation (3.2), due to the lack of periodicity, the impurity now occupies only the two sites x_1 and $x_1 + 2$, and we have the following parametrization:

$$
\begin{aligned}
|3; x_1, \ldots\rangle = \sum_{x_1<x_2} \{&a_1(x_1,x_2)\,|x_1[0], x_2[-]\rangle + b_1(x_1,x_2)\,|x_1[0], x_2[00]\rangle\\
&a_1(x_1,x_2)\,|x_2[-], (x_1+2)[0])\rangle + b_2(x_1,x_2)\,|x_2[00], (x_1+2)[0])\rangle\}.
\end{aligned} \tag{5.9}
$$

The far equations are again

$$\begin{aligned}(E-\beta-\beta^{-1})a_i(x_1,x_2) &= a_i(x_1,x_2-1)+a_i(x_1,x_2+1)\\ &\quad +\beta^{1/2}b_i(x_1,x_2-1)+\beta^{-1/2}b_i(x_1,x_2)\\ (E-1)b_i(x_1,x_2) &= \beta^{1/2}a_i(x_1,x_2+1)+\beta^{-1/2}a_i(x_1,x_2)\qquad i=1,2.\end{aligned}\tag{5.10}$$

For these the ansätze $E=1+\Gamma(\xi_2)\Gamma(\xi_2^{-1})$ together with

$$\begin{aligned}a_1(x_1,x_2) &= \alpha_1^+\Gamma(\xi_2^{-1})\xi_2^{x_2}+\alpha_1^-\Gamma(\xi_2)\xi_2^{-x_2}\\ a_2(x_1,x_2) &= \alpha_2^+\Gamma(\xi_2^{-1})\xi_2^{x_2}+\alpha_2^-\Gamma(\xi_2)\xi_2^{-x_2}\\ b_1(x_1,x_2) &= \alpha_1^+\xi_2^{x_2}+\alpha_1^-\xi_2^{-x_2}\\ b_2(x_1,x_2) &= \alpha_2^+\xi_2^{x_2}+\alpha_2^-\xi_2^{-x_2}\qquad x_2\neq x_1+1\end{aligned}\tag{5.11}$$

solves the far equations.

The near equations are

$$\begin{aligned}&(E-1/\beta)a_1(x,x+1) = a_1(x,x+2)+\beta^{-1/2}b_1(x,x+1)\\ &(E-\beta)a_2(x,x+1) = a_2(x,x)+\beta^{1/2}b_2(x,x)\\ &(E-2)b_1(x,x+1) = (E-2)b_2(x,x)\\ &\qquad =\beta^{1/2}a_1(x,x+2)+\beta^{-1/2}a_1(x,x+1)+\beta^{1/2}a_2(x,x+1)+\beta^{-1/2}a_2(x,x).\end{aligned}\tag{5.12}$$

As usual, we modify our ansatz equation (5.11) to read

$$a_1(x_1,x_1+1)=\alpha_1^+\left(\Gamma(\xi_2^{-1})+\mathcal{F}_{a_1}^+\right)\xi_2^{x_2}+\alpha_1^-\left(\Gamma(\xi_2)+\mathcal{F}_{a_1}^-\right)\xi_2^{-x_2}\tag{5.13}$$

and similar equations for the other amplitudes involving the constants $\mathcal{F}_{a_i}^{\pm}, \mathcal{F}_{b_i}^{\pm}, i=1,2$.

The conditions at the ends are now, depending on the position x_1, given by

$$\begin{aligned}&(E-\beta)a_1(x_1,N) = a_1(x_1,N-1)+\beta^{1/2}b_1(x_1,N-1)\qquad x_1\leqslant N-1\\ &(E-1/\beta)a_2(x_1,1) = a_2(x_1,2)+\beta^{-1/2}b_2(x_1,1)\qquad x_1\geqslant 2\\ &a_1(N-1,N)=0\qquad a_2(0,1)=0.\end{aligned}\tag{5.14}$$

All equations are now satisfied, provided

$$\begin{aligned}&\frac{\alpha_2}{\alpha_1}=-\xi_2^{2N}\qquad \mathcal{F}_{a_1}^{\pm}=-\beta^{1/2}\mathcal{F}_{b_1}^{\pm}\qquad \mathcal{F}_{a_2}^{\pm}=-\beta^{-1/2}\mathcal{F}_{b_2}^{\pm}\xi_2^{\mp 1}\\ &\mathcal{F}_{a_1}^{\pm}=-\beta^{1/2}\xi_2^{\mp 1}\qquad \mathcal{F}_{a_2}^{\pm}=-\beta^{-1/2}\qquad \xi_2^{2N}=1\end{aligned}\tag{5.15}$$

the last one being the BA equation for this case. Again these generate the same eigenvalues as the ones for the XXZ model in the sector with one pseudoparticle.

5.3. The sector r = 4

For this sector, we only outline the procedure. For $\epsilon_i=1, i=1,2$ we make the ansatz

$$a(x_1,x_2)=\alpha_1\Gamma(\xi_1^{-1})\Gamma(\xi_2^{-1})\xi_1^{x_1}\xi_2^{x_2}+\text{permutations}\tag{5.16}$$

and similar equations for $b_i(x_1,x_2), c(x_1,x_2)$. The eight permutations are generated by making the following replacements, in this order: $(\xi_1\to\xi_1^{-1})$, $(\xi_2\to\xi_2^{-1})$, $(\xi_1\to\xi_1^{-1},\xi_2\to\xi_2^{-1})$ and then interchanging $(\xi_1\leftrightarrow\xi_2)$.

The far equations are then solved by

$$\begin{aligned} a(x_1, x_2) &= \alpha_1 \Gamma(\xi_1^{-1})\Gamma(\xi_2^{-1})\xi_1^{x_1}\xi_2^{x_2} + \text{perm} \\ b_1(x_1, x_2) &= \alpha_1 \Gamma(\xi_1^{-1})\xi_1^{x_1}\xi_2^{x_2} + \text{perm} \\ b_2(x_1, x_2) &= \alpha_1 \Gamma(\xi_2^{-1})\xi_2^{x_1}\xi_1^{x_1} + \text{perm} \\ c(x_1, x_2) &= \alpha_1 \xi_1^{x_1}\xi_2^{x_2} + \text{perm}. \end{aligned} \tag{5.17}$$

The equations at the ends $x = 1, x = N-2, N-1$ are then satisfied, provided $\alpha_i = \alpha_{i-2}$, $i = 3, 4, 7, 8$; $\alpha_2 = -\xi_2^{2N}\alpha_1$, $\alpha_2 = -\xi_2^{2N}\alpha_1$, $\alpha_6 = -\xi_2^{2N}\alpha_5$.

As in the periodic case, we have to introduce constants when pseudoparticles are neighbours, as in equations (3.28), in order to satisfy the *near* equations. Their ratios are as in equation (3.30), only the value of $\mathcal{F}_c$ is different. The only yet undetermined coefficient ratio turns out to be

$$\alpha_5/\alpha_1 = -\frac{\xi_2[(1+\xi^{-1})\xi_2 - 2\Delta - E]}{\xi_1[(1+\xi^{-1})\xi_1 - 2\Delta - E]} \tag{5.18}$$

where $\xi = \xi_1\xi_2$. All equations are then satisfied, provided the *BA equations for free boundary conditions* hold:

$$\xi_a^{2N} = \prod_{b=1, b\neq a}^{r} \frac{b(\xi_a^{-1}, \xi_b)}{b(\xi_a, \xi_b)} \qquad a = 1, 2 \tag{5.19}$$

where

$$b(\xi_a, \xi_b) = \frac{\xi_b}{\xi_a}[\xi_b + \xi_a^{-1} - 2\Delta - E][\xi_b^{-1} + \xi_a^{-1} - 2\Delta - E]. \tag{5.20}$$

The block $E = 0$, on the other hand, is again a solution with a high degeneracy. In fact, the state

$$a(x_1, x_2) = \alpha_1 \xi_1^{x_1}\xi_2^{x_2} + \text{permutations} \tag{5.21}$$

and analogous equations for the other amplitudes, is a solution—provided we modify it as usual, when pseudoparticles are neighbours—for any values of ξ_1, $i = 1, 2$ and $\mathcal{F}_c$.

Due to the end conditions, we have more equations to satisfy then in the periodic case. This has the consequence that asymmetric ansätze, like the cases $\epsilon_1 = 1$, $\epsilon_2 = 0$, are not solutions for free boundary conditions and only the cases $\epsilon_1 = \epsilon_2$ remain.

The equation (5.19) admits generalization in analogy with the case of periodic boundary conditions. Thus we again obtain *the same spectrum as the one of the XXZ model* [11], albeit with different multiplicities†.

6. Conclusion

We have applied a modified version of the coordinate BA to the deformed biquadratic Hamiltonian as an interesting example of a simple spin-1 model, where the algebraic BA is not available, although the Yang–Baxter equations are satisfied. The Hamiltonian density is a projector on spin zero and satisfies the Temperley–Lieb algebra. Due to the $U(1)$ invariance, there exists a reference state $|\Omega\rangle$, satisfying $H|\Omega\rangle = 0$. Pseudoparticles are now created on top $|\Omega\rangle$ and the usual machinery is applied. This requires the introduction of 'discontinuous' wavefunctions, which permits us to obtain the eigenstates by a generalization of the usual coordinate BA.

† This has also been noted numerically by F C Alcaraz (private communication).

The ground-state sector $r = N$ of $-H(q)$ for periodic boundary conditions requires an additional modification. If N is even, this sector contains the state $\prod_x |0, x)$. For periodic boundary conditions $-H(q)$ applied to this state creates an extra contribution to the energy arising from the $S_z = 0$ states at the position $x = N$ and $x = 0$. This results in BA equations with changed boundary conditions.

Our spin-1 Hamiltonian is Hermitian only for $q + q^{-1} > 2$ and outside this region complex eigenvalues arise. Nevertheless, the block $\epsilon_i = 1$, which contains the ground state of the complete antiferromagnetic model, constitutes by itself a perfectly unitary theory with real eigenvalues. Therefore, the model, which is not unitary, does nevertheless possess a subset of eigenvalues, belonging to a unitary one, although the corresponding states are not decoupled from the rest. A more detailed investigation of this point should also shed light on similar situations in other settings [12].

Our results are in agreement with the rather extensive numerical calculations performed by Alcaraz and Malvezzi [13], who, in particular, observed that $(-H)$ and the spin-$\frac{1}{2}$ XXZ Hamiltonian share the same ground state for N = odd.

For free boundary conditions the spectrum is, up to multiplicites, identical to the one of the XXZ-model. We also verified that our BA equations for free boundary conditions generate highest-weight states of the quantum group $U_q sl(2)$. For $E = 0$ this is not always true.

There are several issues left for future work. In particular, one would like to clarify from an algebraic point of view the equality of the spectra of the biquadratic and the XXZ models for free boundary conditions and the inclusion of the XXZ spectrum in the one of the biquadratic Hamiltonian for periodic boundary conditions. Furthermore, the completeness and complete characterization as highest-weight states of the BA eigenstates, has been left open.

Finally it is clear that the BA equations have to be proved only for two excitations, since our model is derived from a Yang–Baxter equation. The validity of BA for the higher sectors should follow automatically. We have no explicit proof of this statement.

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